MEMORANDUM

TO: File
FROM: Jordan Bleicher
RE: Meeting with Bloomberg LP
DATE: September 16, 2010

On August 23, 2010, representatives from the Securities and Exchange Commission ("SEC") met with representatives from Bloomberg LP ("Bloomberg") and Williams & Jensen, PLLC ("Williams & Jensen") at the SEC's headquarters in Washington, DC. The SEC representatives were Jordan Bleicher, Henry Hu, Bruce Kraus, and Harvey Westbrook. The Bloomberg representatives were Gregory Babyak and Adam Litke. The Williams & Jensen representative was Joel Oswald. Bloomberg and Williams & Jensen discussed the collection of data needed to measure systemic risk and urged the SEC to support an open source approach to the development of party and financial instrument identifiers.

On August 30, Adam Litke sent the attached materials to provide additional information about topics discussed.

Agenda - August 23, 2010 (12:30pm)

I. Systemic Risk

II. The Functioning of the Office of Financial Research

From:	Hu, Henry
To:	Bleicher, Jordan
Subject:	FW: Follow up to our meeeting last week
Date:	Tuesday, August 31, 2010 8:25:21 AM
Attachments:	bsym-whitepaper.pdf bloomberg_research_report_cast.pdf eisenberg noe_systemic risk in financial networks.pdf

-----Original Message-----
From: ADAM LITKE, BLOOMBERG/ 731 LEXIN [mailto:alitke2@bloomberg.net]
Sent: Monday, August 30, 2010 4:48 PM
To: Hu, Henry; Kraus, Bruce; Westbrook, Harvey B.
Cc: gbabyak@bloomberg.net; jgoswald@wms-jen.com; defranasiak@wms-jen.com
Subject: Follow up to our meeeting last week

Dear Sirs:

Thank you very much for taking the time to meet with us last Monday.

As I stated in the meeting, we believe that there are several key information areas where the interests of the financial services industry and the regulators converge. To the extent that all of the relevant agencies, SEC, Treasury, OFR, FRB, CFTC, FDIC adopt common standards for the format of the data that they are collecting from the firms they regulate, it will make it easier and cheaper for these firms to comply with such requests. It should also serve as an incentive towards internal data standardization for those firms that have not fully integrated their own data.

In addition to the public standards such as FPML and XBRL, which we fully support, there are several areas where no public standard yet exists. The most important of these are security identifiers and capital structure. As promised, I am attaching information on BSYM and CAST.

BSYM is Bloomberg's open symbology for security identifiers. It was designed to deal with the many different symbol systems used in different securities markets around the world and to solve some of the problems, such as re-use of symbols, that are inherent in some of these systems. We have made this available in the public domain. Full information on BSYM is available on the website http://bsym.bloomberg.com. I have also attached a white paper.

CAST is Bloomberg's capital structure product. In practice it is a combination of two types of product. The first is a complete list of public companies and their subsidiaries. At the present time, this list only includes entities that have, at some point in their existence, had either bonds, loans, or equity shares outstanding . However, as this information is reconciled back to the filings of these companies, it would be a relatively simple matter for us to include all subsidiaries. The information is stored in tree form and shows guarantee relationships between the legal entities. The second type of data is the outstanding obligations of these entities as well as considerable detail on this issuance including covenant information. CAST presently covers almost 15,000 companies around the world including over 97% of US companies. I have enclosed a short document that shows how CAST works. In a few days, I will also forward a larger document that gives more details. If you are interested in this product, including how we maintain the accuracy of our data, I would be more than happy to put you in touch with Liz Goldenberg, the product manager for CAST.

Finally, I have attached a copy of the paper by Eisenberg and Noe on systemic risk and market clearing. I think you may find it of interest. Eisenberg is presently at work on a multi-period sequal to the paper and I would be more than happy to put you in touch with him if you want to discuss his work. directly.

Please feel free to contact me if you should have questions about any of this.

Adam Litke

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BSYM IDENTIFIERS ADVANCE CAUSE OF OPEN, AUTOMATED SECURITIES TRADING

Bloomberg Open Symbology Introduces New Operational Efficiencies and Cost Reductions for the Trading Community

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INTRODUCTION

Chaos theory has nothing on the complexity generated everyday by the millions–perhaps billions–of security transactions that cross trading floors, clearinghouses and exchanges all over the world. Almost every aspect of securities management is based on closed systems that use proprietary identifiers that are privately owned and licensed. Closing each deal is as much an exercise in translation as it is in transaction processing, as traders, investors and brokers wrestle with multiple proprietary formats to determine what a security is, who owns it, how much it is worth, and when the deal should be closed. It introduces a tremendous amount of friction into the trade lifecycle and creates opaqueness where clarity is sought. In addition, the use of proprietary identifiers adds significant cost and overhead when users wish to integrate data from disparate sources or migrate to a different market data system.

Symbols are essential to the securities industry. Each one uniquely identifies a specific security instrument, just as a VIN number uniquely identifies every motor vehicle. Symbols are used to research and trade securities, assess risk, manage portfolios, and manage settlement and clearing.

Even though there are national numbering agencies that create unique identifiers, symbol sets have evolved in complexity over the years to match the growing sophistication of the products they describe. Sets must be extended and created to catalog levels of granularity in symbology that a single ID simply can't capture. For privately traded, over-the-counter products, there may be no proprietary ID available.

The evolution of advanced symbologies has helped the securities industry grow, but the limitations and costs imposed by the closed systems have become more apparent as companies and institutions continue to integrate operations on a global scale. Proprietary symbology now stands as one of the most significant barriers to increased efficiency and innovation in an industry that sorely needs it. Moreover, the lack of common identifiers is a key roadblock to achieving the holy grail of straight-through processing (STP).

CONSIDER THE FOLLOWING:

- Licensing fees require firms to pay for each symbol system they use. International firms bear an especially heavy burden, because they often have to license several symbologies in order to manage trading operations in several countries.
- Restrictions imposed by proprietary symbologies prevent companies from easily mapping one set of codes to another. This hinders integration of market data from diverse sources as well as efforts to automate trade and settlement activities.
- Market data consumers who adopt proprietary symbols for use in their own systems must not only pay licensing fees, but such symbols also lead to significant future costs associated with efforts to connect to emerging trading systems.
- Proprietary trading environments may have worked well for years; but they are a byproduct of a time when data systems operated largely as islands that did not have to interoperate with other systems.

Current trends dictate a different approach. Markets, customers and governments are demanding greater connectivity, transparency and efficiency. What's more, the openness of Internet-based systems has profoundly altered the way businesses–and individuals–collect, manage and share information. Thus, in addition to new regulations that demand clarity and accountability, the move to open symbology is being driven by growing investor and institutional demands.

Adopting an open system of shared symbology establishes the foundation for a tremendous leap forward in the efficient trade and settlement of securities. Such a system will allow firms and technology service providers to shift resources from laborious, inefficient processes to new investments in tools and products that will better serve clients.

An open system answers the call for greater transparency. Eliminating the need to remove proprietary IDs and re-map securities will greatly simplify the steps needed to migrate between market data platforms and trading systems. Availability of a central symbology reference will facilitate mapping between users' internal systems and create opportunities for integration and automation of the global enterprise.

Introducing Bloomberg Open Symbology (BSYM)

In response to the market demand for open systems and symbology, Bloomberg has released Bloomberg Open Symbology ("BSYM") identifiers and has dedicated these identifiers to the public as set forth at the BSYM web site (bsym.bloomberg.com). BSYM is now available as a non-proprietary, open, security identification system that anyone can adopt. BSYM offers any company involved in securities trading a number of advantages over closed and costly systems.

- BSYM is a universal securities symbology that offers companies the potential to streamline internal management functions and reduce costs associated with maintaining multiple symbology systems.
- BSYM can be used independently of any Bloomberg product or system and there is no limit on the term of the license, so users will never be required to pay for or remove BSYM identifiers from their systems.
- BSYM can be applied in many unique ways. For example, a middleware tool built on BSYM would create a bridge between companies using proprietary systems, allowing them to speak in a common language without the need to license additional symbologies. This creates significant cost savings through reduced licensing fees and automated processing for all participating firms.
- BSYM can be used for any purpose and incorporated into any system now and in the future. Systems built on BSYM symbology will never be required to pay licensing fees for its use.
- BSYM will greatly reduce the cost associated with changing platforms, allowing companies the freedom to select systems that best suit their needs.

The call from the market for systems that encourage innovation and efficiency couldn't be clearer. Bloomberg is committed to delivering the tools and standards that will help the securities industry enjoy a new era of advancement.

Using BSYM Identifiers

BSYM is not a single identifier. It is the name for Bloomberg's family of security identifiers. The BSYM identifiers allow trading and market data systems to cross reference security identifiers from various sources and various Bloomberg data products. Toward that end, Bloomberg is allowing BSYM Identifiers to be freely reproduced, distributed, transmitted, used, modified, built upon, or otherwise exploited by anyone for any purpose at no cost. Indeed, Bloomberg is encouraging all members of the trading community to use BSYM identifiers for integration and redistribution within and beyond their organizations.

BSYM identifiers available at bsym.bloomberg.com can be used to map data across all of Bloomberg's raw data products, and they can also be used to determine the 'parse key' for loading a security on the Bloomberg Terminal command line. BSYMs can be searched by many proprietary IDs, such as the Stock Exchange Daily Official List (SEDOL), Committee on Uniform Securities Identification Procedures (CUSIP), and the International Securities Identification Number (ISIN), as well as by security description, security type, and pricing source. Bsym.bloomberg.com also provides predefined dump files and searches, as well as custom search and filter capability. All data is refreshed on a daily basis.

Expanded security coverage, additional Bloomberg Identifiers, and additional mechanisms for searching and requesting data will be added as needs are defined.

For the purpose of BSYM, a security is defined as an issue that may be priced by multiple pricing sources–such as IBM's common stock, and the 10 Year US Treasury Bond. An Instrument is defined as a security that is pricing or trading in a specific venue – such as NYSE, AMEX, US Composite, a specific broker-dealer, or Bloomberg Generic Pricing. Instruments are identified at the level of their market price (Ticker + Pricing Source, BSID). Securities are identified at the level of the issue itself (Unique Identifier, Name).

UNDERSTANDING BSYM FIELDS

Name Name of the company or brief description of the security. The Name of an instrument may change in conjunction with corporate actions.

Unique ID Unique identifier assigned by Bloomberg to all securities. This id can be used for mapping B-Pipe and API identifiers to Bloomberg's Data License products. Data License provides extensive fundamental and security master data that complements Bloomberg's real time data offerings. The Unique ID can also be used to load a security onto the Bloomberg Terminal by prefixing the value with 'ID' on the command line. In general, for equities, the Unique "ID" groups together instruments that contribute pricing to the same composite market (e.g. US, JP). However, an exception to this rule occurs when the same security trades in different currencies in the same market, rendering the trading instruments nonfungible. In these cases, securities will have a different Bloomberg Ticker and Unique ID for each currency in which the security trades. MiFID OTC markets are a good example of this (see examples below). For fixed income securities, the Unique ID identifies a security across all dealers and currencies, so the Unique ID is not an indicator of fungibility or participation in a composite for fixed income. The Unique ID of a security may change in conjunction with corporate actions.

Security Type Description of the specific security type within its Bloomberg market sector (Yellow Key). This classification corresponds to the predefined list of files that are available on the BSYM website. Mappings from Market Sector to Security Type are available on the Web site under the 'Security Type Mapping' link.

Market Sector Market sector that Bloomberg has assigned to the security. This corresponds to the Bloomberg Yellow Key.

Pricing Source Acronym or short code for the market data source, used on the B-Pipe feed. This field provides B-Pipe source codes for a variety of asset types. Note that in some cases the source in this field is not loadable on the terminal. B-Pipe makes a distinction between sessions in the source field, while the terminal handles this by means of the PCS <GO> function, which allows configuration of user-specific session preferences. In general, for markets that have electronic, pit and combined sessions, the pit session will use the source code found on the terminal: the electronic session will use that code prefixed by "e", and the combined session will use the source code prefixed by "c". In addition, B-Pipe assigns a source code for indices that are not used on the terminal, e.g. DJI for Dow Jones pricing the INDU index. Pricing Source is currently available only for B-Pipe priced securities. See the "Pricing Source Descriptions" link on the BSYM Web site for a mapping of the Pricing Source code to a description of the source.

BSID (Bloomberg Security ID Number with Source) - Unique integer identifier for all B-Pipe securities. This identifier is used for subscription services in B-Pipe (Managed and On Demand). BSIDs are unique at instrument level and have a 1-many mapping with the Unique ID field described above. The BSID of a security may change in conjunction with corporate actions and is available only for B-Pipe priced securities.

Ticker Unique B-Pipe ticker symbol. Combined with the Pricing Source and Market Sector, this forms a loadable security string on the Bloomberg terminal for most securities. The ticker may change in conjunction with corporate actions and is available only for securities that are priced on B-Pipe.

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BSYM TICKER CREATION

The rules for forming the BSYM Ticker vary according to security class.

CURRENCY	For basic cross rates B-Pipe uses the ISO codes for both currencies. Due to the wide variety in type and the desire to keep them short currency derivative symbols are not so clear-cut. They tend to be based on the ISO codes, but they are often truncated. They can also be based on the futures exchange codes such as ED, BP, SF, etc. The type of derivative is often included as an abbreviation, and although the use of the abbreviation is consistent for the derivative it is difficult to predict. Time periods are almost always included when relevant. Spot (implied cross rate against the US dollar) currencies are not available. Instead you have to ask for explicit cross rate, e.g., instead of using "EUR" or "JPY" use "EURUSD" or "JPYUSD".
EQUITY	Equity symbols are usually the exchange ticker.
FIXED INCOME	Fixed income symbols are built by combing a root symbol, the coupon, the maturity date and an optional series. A zero coupon is represented by "0", e.g., "PEISTP 0 01/29/23" A floating coupon is represented by "F", e.g., "CNC F 12/04/13" A variable coupon is represented by preceding the coupon with "V", e.g., "MQB V5.75 02/18/13 1" Loans are represented by "L" I in the coupon position, e.g., "C L 05/01/98" A perpetual instrument is represented by preceding a pseudo maturity date with "P", e.g., "BMO 5.474 P12/29/49 D". The month, day and year of maturity dates are always two digits, 0 padded if needed. Pseudo maturity dates are often "12/29/49".
FUND	For exchange traded funds the symbol is usually the exchange ticker. Other fund symbols are mnemonics or acronyms built from the description of the index.
FUTURE	Future symbols are based on the exchange ticker. Physical, financial and currency futures symbols use a one, two or three character root for the commodity followed by the standard month letter and single last digit of the year. If the commodity code is a single character, such as "W", it is padded with a space so that it is always two characters
INDEX	Major exchange indices usually use the common symbol, but the source is not always obvious, e.g., B-Pipe subscriptions "/DJI/INDU Index", "/OPRA/SPX Index", "/JT/TPX100 Index". Other index symbols are mnemonics or acronyms built from the description of the index.
OPTION	For future options use the ticker of the underlying future with a "P" or "C" appended, a space and then the strike price, e.g., "CDM8C 99.5" For equity options use the ticker of the underlying equity, a space followed by the month or month/day of expiration, a space and then a "C" or "P" with the strike price appended. E.g. "IBM 10 C140". All strike prices drop trailing 0s and decimal points, e.g. ,"15.15" is "15.15" but "15.10" is "15.1" and "123.00" is "123".
WARRANT	For listed warrants with an official exchange symbol, the exchange symbol is used. For warrants that are not listed, or that do not have an official symbol, the value is algorithmically generated by Bloomberg using rules that vary by issuer.

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Cross Referencing Field Names across Products

BSYM identifiers can be used to map real time data from B-Pipe or Bloomberg's Server API to Bloomberg's Data License reference data and corporate actions products.

Field ID (FLDS)	Field Mnemonic (DL / API)	B-Pipe Field	BSYM Field
DS002	NAME	Reference.Security.ID.Name	Name
ID059	ID_BB_UNIQUE	Reference.Security.Bloomberg.UniqueID	Unique ID
DS213	SECURITY_TYP	Reference.Security.Type	Security Type
DS122	MARKET_SECTOR_DES	NA*	Market Sector
DY003	ID_BB_SEC_NUM_DES	Reference.Security.ID.Bloomberg.Symbol	Ticker
DX282	FEED_SOURCE	MD.Source	Pricing Source
ID122	ID_BB_SEC_NUM_SRC	MD.Security.ID.BSID	BSID

*Reflected in the message type of the B-Pipe reference data message

EXAMPLES OF BSYM IDENTIFIERS ACROSS MARKETS AND SECURITY CLASSES

Equity (Single Currency per Listed Market):

Column Heading	IBM (US Composite)	IBM (NYSE)	IBM (German Composite)	IBM (Berlin Exchange)
Name	INTL BUSINESS MACHINES CORP	INTL BUSINESS MACHINES CORP	INTL BUSINESS MACHINES CORP	INTL BUSINESS MACHINES CORP
Unique ID	EQ0010080100001000	EQ0010080100001000	EQ0010080100001007	EQ0010080100001007
Security Type	Common Stock	Common Stock	Common Stock	Common Stock
Market Sector	Equity	Equity	Equity	Equity
Ticker	IBM	IBM	IBM	IBM
Pricing Source	US	UN	GR	GB
BSID	399432473346	627065740034	395137622225	1623498268881

Equity (MiFID On Book):

Column Heading	LLOY (London Listed - GBp)	LLOY (PLUS - GBp)	LLOY (Chi-X - GBp)
Name	LLOYDS BANKING GROUP PLC	LLOYDS BANKING GROUP PLC	LLOYDS BANKING GROUP PLC
Unique ID	EQ0011242800001000	EQ0000000005037071	EQ0000000002865282
Security Type	Common Stock	Common Stock	Common Stock
Market Sector	Equity	Equity	Equity
Ticker	LLOY	LLOY	LLOY
Pricing Source	LN	PZ	IX
BSID	678605350662	1997163581670	2005750482138

Equity (MiFID OTC):

Column Heading	LLOY (London OTC - GBP)	LLOY (Chi-X OTC - GBP)	LLOY (London OTC - Euro)	LLOY (Chi-X OTC - Euro)
Name	LLOYDS BANKING GROUP PLC	LLOYDS BANKING GROUP PLC	LLOYDS BANKING GROUP PLC	LLOYDS BANKING GROUP PLC
Unique ID	EQ0000000005002180	EQ0000000005002180	EQ0000000005103164	EQ0000000005103164
Security Type	Common Stock	Common Stock	Common Stock	Common Stock
Market Sector	Equity	Equity	Equity	Equity
Ticker	LLOYGBP	LLOYGBP	LLOYEUR	LLOYEUR
Pricing Source	XJ	XC	XJ	XC
BSID	6111741723983	6064497083727	6111741807243	6064497166987

Commodity/Index Future (With Sessions):

Column Heading	LCJ0 PIT (CME)	LCJ0 ELEC (CME)	LCJ0 COMB (CME)
Name	LIVE CATTLE FUTR Apr10	LIVE CATTLE FUTR Apr10	LIVE CATTLE FUTR Apr10
Unique ID	IX8013948-0	IX8013948-0	IX8013948-0
Security Type	Physical commodity future.	Physical commodity future.	Physical commodity future.
Market Sector	Comdty	Comdty	Comdty
Ticker	LCJ0	LCJ0	LCJ0
Pricing Source	CME	eCME	cCME
BSID	614188830019	2078772677955	9981512502595

Single Stock Future:

Column Heading	Daimler May 2010 (LIFFE)	Daimler May 2010 (Eurex)	Daimler May 2010 (Milan)
Name	DaimlerChrysler AG May10	DaimlerChrysler AG May10	DaimlerChrysler AG May10
Unique ID	EF12666074700074186777	EF12399952420074448897	EF12666075260074186844
Security Type	SINGLE STOCK FUTURE	SINGLE STOCK FUTURE	SINGLE STOCK FUTURE
Market Sector	Equity	Equity	Equity
Ticker	DCX=K0	DCX=K0	BDCX=K0
Pricing Source	LIF	EUX	SM
BSID	609900809283	279182846813	476756828955

Corporate Bond:

Column Heading	GS 7.5 02/15/19 (TRACE)	GS 7.5 02/15/19 (German Composite)	GS 7.5 02/15/19 (CBBT)
Name	GOLDMAN SACHS GROUP INC	GOLDMAN SACHS GROUP INC	GOLDMAN SACHS GROUP INC
Unique ID	COEH7068206	COEH7068206	COEH7068206
Security Type	GLOBAL	GLOBAL	GLOBAL
Market Sector	Corp	Corp	Corp
Ticker	GS 7.5 02/15/19	GS 7.5 02/15/19	GS 7.5 02/15/19
Pricing Source	TRAC	GR	CBBT
BSID	631369281816	395146080536	665729020184

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Mortgage:

Column Heading	FNCL 4 4/10 (Bloomberg Generic)	FNCL 4 4/10 (Composite Bloomberg Bond Trader)
Name	FNCL 4 4/10	FNCL 4 4/10
Unique ID	MG%3278ACK	MG%3278ACK
Security Type	MBS 30yr	MBS 30yr
Market Sector	Mtge	Mtge
Ticker	FNCL 4 4/10	FNCL 4 4/10
Pricing Source	BGN	CBBT
BSID	12894812880	665729841872

Preferred:

Column Heading	RBS CAPITAL FND TRST VII (US Composite)	RBS CAPITAL FND TRST VII (NYSE Preferred)	RBS CAPITAL FND TRST VI (US Composite)	RBS CAPITAL FND TRST VI (NYSE Preferred)
Name	RBS CAPITAL FND TRST VII	RBS CAPITAL FND TRST VII	RBS CAPITAL FND TRST VI	RBS CAPITAL FND TRST VI
Unique ID	PFEP0109264	PFEP0109264	PFEP0093955	PFEP0093955
Security Type	PUBLIC	PUBLIC	PUBLIC	PUBLIC
Market Sector	Pfd	Pfd	Pfd	Pfd
Ticker	ABNA 6.08 P12/31/49 G	ABNA 6.08 P12/31/49 G	ABNA 6.25 P12/31/49 F	ABNA 6.25 P12/31/49 F
Pricing Source	US	SNY1	US	SNY1
BSID	399432238302	618475570398	399432238212	618475570308

Government Bonds:

Column Heading	T 7.5 11/15/24 (Standard Chartered)	T 7.5 11/15/24 (Citigroup)	T 7.5 11/15/24 (Credit Suisse)
Name	US TREASURY N/B	US TREASURY N/B	US TREASURY N/B
Unique ID	*	*	*
Security Type	US GOVERNMENT	US GOVERNMENT	US GOVERNMENT
Market Sector	Govt	Govt	Govt
Ticker	T 7.5 11/15/24	T 7.5 11/15/24	T 7.5 11/15/24
Pricing Source	SCBX	CGUK	CSFB
BSID	12004433733107	5063766582771	5059471615475

* In some cases Unique ID is blank due to the value being based on a proprietary ID. New IDs are being assigned and will be updated soon.

Currency:

Column Heading	USD-EUR X-RATE (Tokyo Composite)	USD-EUR X-RATE (New York Composite)	USD-EUR X-RATE (CBA Bank)	USD-EUR X-RATE (8am Fixing Rate)
Name	USD-EUR X-RATE	USD-EUR X-RATE	USD-EUR X-RATE	USD-EUR X-RATE
Unique ID	IX430979-0	IX430979-0	IX430979-0	IX430979-0
Security Type	CROSS	CROSS	CROSS	CROSS
Market Sector	Curncy	Curncy	Curncy	Curncy
Ticker	USDEUR	USDEUR	USDEUR	USDEUR
Pricing Source	CMPT	CMPN	CBAX	F080
BSID	425201809525	416611874933	12648678733941	14289356241013

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Index Option:

Column Heading	June 10 Puts on SPX (US Composite)	June 10 Calls on SPX (US Composite)
Name	June 10 Puts on SPX	June 10 Calls on SPX
Unique ID	IX6956513-0-1400	IX6956509-0-9400
Security Type	Index Option	Index Option
Market Sector	Index	Index
Ticker	SPX 06/19/10 P800	SPX 06/19/10 C800
Pricing Source	US	US
BSID	399438708018	399438708043

Equity Option:

Column Heading	April 10 Calls on VOD US (US Composite)	April 10 Calls on VOD US (AMEX)	April 10 Calls on VOD LN (LIFFE)	April 10 Calls on VOD NQ (EUREX)
Name	April 10 Calls on VOD US	April 10 Calls on VOD US	April 10 Calls on VOD LN	April 10 Calls on VOD NQ
Unique ID	EO1016052010040181900001	EO1016052010040181900001	EO101605201004038DC00006	EO101605201004028F000002
Security Type	Equity Option	Equity Option	Equity Option	Equity Option
Market Sector	Equity	Equity	Equity	Equity
Ticker	VOD 04/17/10 C12.5	VOD 04/17/10 C12.5	VOD 04/16/10 C110	VOD 04/16/10 C120
Pricing Source	US	UA	LIF	EUX
BSID	399444460111	523998511695	609900043157	279187460380

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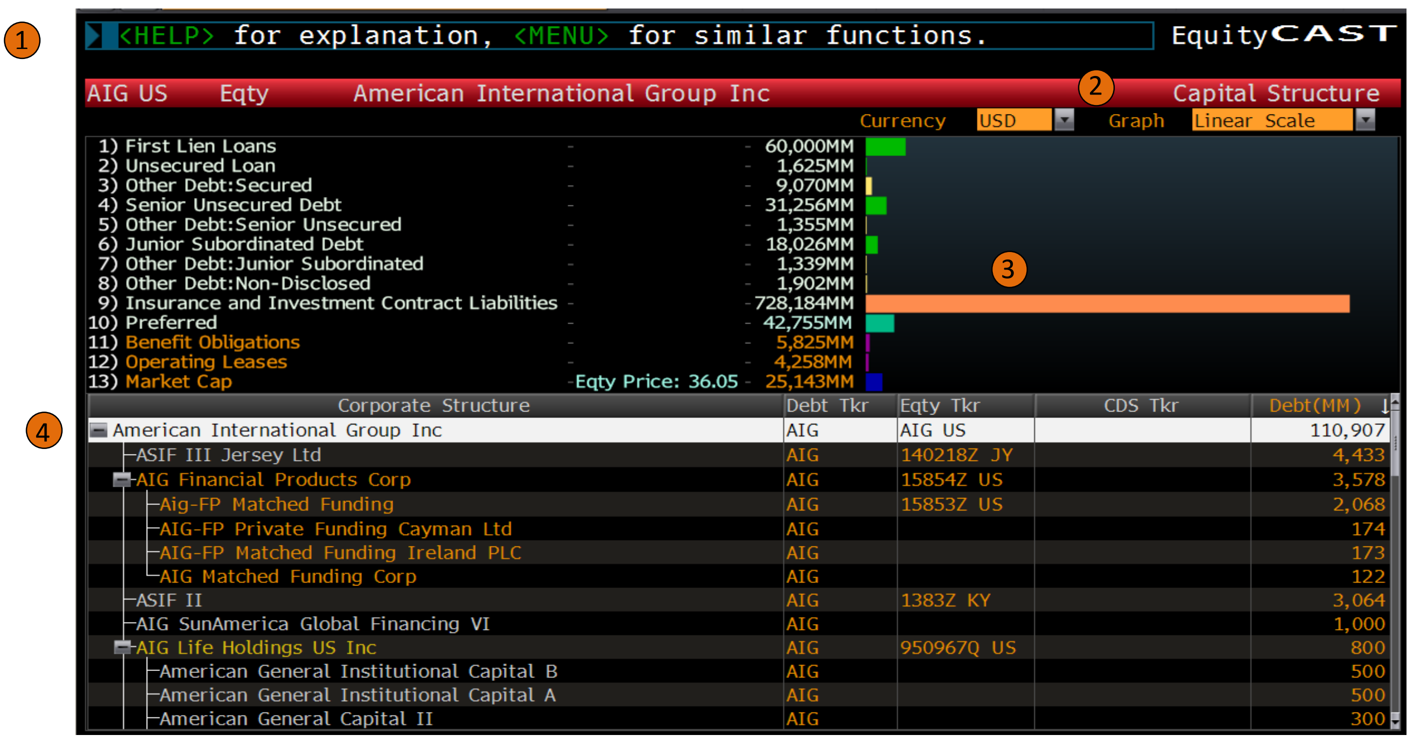
CAST <GO>

CAPITAL STRUCTURE ON BLOOMBERG

Use CAST <GO> to display a selected company's liabilities and the amount owed to investors at each level of the company's capital structure.

This provides transparency into how creditors may be paid in the event of the company's bankruptcy. Typically, secured lenders are paid first, followed by senior unsecured lenders, subordinated lenders, junior subordinated lenders, and finally, shareholders. CAST also displays information on the corporate structure of the company, such as the subsidiaries.

HOW TO ANALYZE CAPITAL STRUCTURE



1. Choose an applicable security and enter CAST <GO>.

2. To filter the debt by currency, choose the appropriate option from the amber dropdown to the right of CURRENCY. To display the graph with a Log Scale or Linear Scale, choose the appropriate option from the amber dropdown to the right of GRAPH.

3. To view the underlying individual securities or as-reported financial data, click on the data label or the accompanying shaded bar.

4. Additional elements of the company's corporate structure are displayed in the lower half of the screen. Clicking on [-] or [+] to the left of the appropriate category will collapse or expand each category.

Display Security Detail Information

Clicking on a data label or shaded bar on the Capital Structure screen will open either a Security Detail Screen or FA<GO>.

FA<GO> from CAST will display the company's financial history, indentify trends and gain data transparency to assist in analyzing the value of a potential investment, partnership, or acquisition.

How to Interpret Bar Chart Colors

Green: Debt included in the BLOOMBERG PROFESSIONAL® service's security database.

Yellow: Debt disclosed in company filings, but not included in Bloomberg's security database. The data displayed has been confirmed by the company.

Orange: Insurance and investment contract liabilities that are disclosed in filings.

Aqua: Preferred shares.

Purple: Data that comes from financial filings, such as operating and/or capital leases, deposits, pension obligations, and accounts payable.

Pink: Municipal debt.

Blue: Represents the market capitalization as of the previous day's close.

Violet: Represents asset-backed and mortgage-backed debt that is part of Bloomberg's security database.

Systemic risk in financial networks

by **Larry Eisenberg**[1] **and Thomas H. Noe**[2]

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Summary. We consider default by firms that are part of a single clearing mechanism. The obligations of all firms within the system are determined simultaneously in a fashion consistent with the priority of debt claims and the limited liability of equity. We first show, via a fixed-point argument, that there always exists a "clearing payment vector" that clears the obligations of the members of the clearing system; under mild regularity conditions, this clearing vector is unique. Next, we develop an algorithm that both clears the financial network in a computationally efficient fashion and provides information on the systemic risk faced by system firm. Finally, we produce qualitative comparative statics for financial networks. These comparative statics imply that, in contrast to single-firm results, unsystematic, nondissipative shocks to the system will lower the total value of the network and may *lower* the value of the equity of some of the individual network firms.

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Systemic risk in financial networks

1 Introduction

One of the most pervasive aspects of the contemporary financial environment is the rich network of interconnections among firms. Although financial liabilities owed by one firm to another are usually modeled as unidirectional obligations dependent only on the financial health of the issuing firm, in reality, the liability structure of corporate obligations is invariably much more intricate. The value of most firms is dependent on the payoffs they receive from their claims on other firms. The value of these claims depends, in turn, on the financial health of yet other firms in the system. Moreover, linkages between firms can be cyclical. A default by firm A on its obligations to firm B may lead B to default on its obligations to C. A default by C may, in turn, have a feedback effect on A. Thus, financial system architectures may exhibit cyclical dependence in interfirm obligations. We consider the problem of finding a clearing mechanism in cases in which this sort of cyclical interdependence is present.

All markets have some kind of clearing mechanism. Perhaps clearing mechanisms for inter-bank payments and for listed exchanges have received the most attention. In the United States, for example, CHIPS and Fedwire are the main banking clearing mechanisms; in Germany, the Abrech-nung and the EAF (Elektronische AI) rechnung mit Filetransfer) performs this function. Regarding clearing mechanisms, one of the attractions of trading on a listed options exchange, the CBOE for example, is that the Options Clearing Corporation is the counterparty to every trade. Hence credit considerations do not prohibit lower credit traders from participating in these markets. These pay-ment systems are forced to confront large defaults on a regular basis. Examples of such defaults include the failure of I.D. Herstatt in 1974 and the Bank of New York overnight shortfall of $22.6 billion dollars in 1985. System-wide meltdowns also occur. For example, consider the collapse of the Tokyo real estate market, the bankruptcy and public bailout of American S&Ls to the cost of about $500 billion dollars and the Venezuelan bank crisis of 1994. One of the most interesting failures of a tightly interconnected clearing system was the 1982 collapse al-Manakh Stock Market in Kuwait.

The clearing system, consisting of approximately 29,000 post-dated checks written by traders, collapsed after a 40 percent drop in market values. The nominal gross liabilities of the participants in the market to each other at the time of the collapse was more than four times Kuwait's GDP (Eliman, Girgis, and Kotob, 1997).

Surprisingly, despite the obvious importance of the "architecture of financial linkages" for determining the return-generating process for financial assets, little has been written on cyclical financial interconnections. Bilateral clearing has been throughly analyzed in Duffie and Huang (1996). Rochet and Tirole (1996) analyzed the incentive and monitoring impact of an interbank loan. From a more empirical perspective, Angelini and Russo (1996) develop an empirical model of intercorporate defaults. In this model, the probability that a default by one firm triggers another firm's default is exogenously specified without modeling intercorporate cash flows. Eliam, Girgis, and Kotob (1997) report the actual procedure used to clear intercorporate debts after the Kuwaiti stock market crash. However, to our knowledge, this paper is the first to analyze, in a general fashion, the properties of intercorporate cash flows in financial systems featuring cyclical obligations and endogenously-determined clearing vectors.

This lack of attention to cyclicality is even more surprising given the extensive literature modeling default in a simple unidirectional and bilateral context. In fact, the whole literature on term-structure of interest rates ignores the considerations mentioned above. While modeling the valuation of a firm's debt as independent from that of other firms simplifies debt and equity models, this assumption becomes questionable in portfolio management, corporate bond trading and the analysis of counterparty credit risk. The aim of this paper is to investigate the propagation of risk through clearing systems and the effects of this risk propagation on the return-generating process of system-participants. A desideratum for the future development of these lines of research is the development of a simple, tractable model for computing clearing vectors for intralinked financial systems. The aim of this paper is to provide such a model.

We develop a fairly general model of a clearing system. The model satisfies the standard conditions on clearing vectors imposed by bankruptcy law: proportional repayments of liabilities in default, limited liability, and absolute priority. We shall show, via a fixed-point argument, that there always exists a "clearing payment vector," consistent with these conditions, that specifies the payment made by each node in the system. Moreover, under mild regularity conditions, this clearing vector is unique and may be characterized in two ways. First, it is the limit of a finite sequence of clearing vectors produced by "fictitious sequential default" algorithm. This algorithm, as well as quickly yielding the clearing vector, produces a natural metric for examining the systemic risk exposure of firms in the financial system. Second, the clearing vector maximizes the weighted average of firm payments regardless of weighting scheme. Our results demonstrate that any clearing payment vector maximizes both the cents-on-the-dollar repaid and the total repayments to creditors.

After analyzing the clearing vector, we perform comparative statics on the clearing payment vector, determining the nature of its dependence on the vector of exogenous cash infusions as well as on the architecture of financial liabilities linking the various members of the system. More specifically, we show that the clearing payment vector is a multidimensional concave function of operating cash flows and the level of nominal payments, and that the value of equity is not generally convex in cash flows. These results imply that the total value of firms in the system is concave in exogenous cash flows. In turn, this increased concavity implies that increased volatility, by lowering expected interfirm payments, will lower the total value of nodes in the system, even though there are no costs to insolvency in our model. (And thus the real economic effect of such a shock is nil.) Our results suggest that using changes in total asset values to measure the effect of an economic shock on a group of tightly interconnected companies (e.g., Japanese banks) can be highly misleading.

The paper is organized as follows. In Section 2, we present the model and develop the basic machinery, including existence uniqueness results. In Section 3, we present the two characterizations of the clearing vectors and examine their consequences. In Section 4, we derive comparative statics of the clearing system. Section 5 concludes the paper and considers some extensions.

2 Framework and basic results

2.1 Preliminaries

Let $\Re^n$ represent n-dimensional Euclidean vector space. Let $\mathcal{N} = \{1, 2, ...n\}$. For any two vectors x, $y \in \Re^n$, define the lattice operations

$$x \wedge y := (\min[x_1, y_1], \min[x_2, y_2] \ldots \min[x_n, y_n])$$

$$x \vee y := (\max[x_1, y_1], \max[x_2, y_2] \ldots \max[x_n, y_n]).$$

Let $\mathbf{1}$ represent an n-dimensional vector, all of whose components equal 1, i.e., $\mathbf{1} = (1, \ldots, 1)$. Similarly, let $\mathbf{0}$ represent an n-dimensional vector, all of whose components equal 0. Let $\|\cdot\|_1$ denote the ℓ^1-norm on $\Re^n$. That is, for each $x \in \Re^n$ let

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|.$$

For each $n \times n$ matrix, M, let $\rho(M)$ represent the spectral radius of the matrix, the eigenvalue of the matrix of maximal absolute value. With each linear transform defined on $\Re^n$ there is an associated $n \times n$ matrix M. Let $|||\cdot|||_1$ be the operator matrix Norm associated with $||\cdot||_1$. That is, for each $n \times n$ matrix, define

$$|||M|||_1 \equiv \operatorname*{Sup}_{\|x\|_1 \le 1} ||Mx||.$$

It is well known that [e.g., Horn and Johnson (1985) §5.6.4, page 294] that, for any $n \times n$ matrix M, we have

$$|||M|||_1 = \operatorname*{Max}_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

An important definition for our future analysis is of a non-expansive map. A map $T: \Re^n \to \Re^n$ is (ℓ^1)-*nonexpansive* if, $\forall x \in \Re^n$,

$$\|T(x) - T(y)\|_1 \le \|x - y\|_1.$$

Whenever an ordering of elements of $\Re^n$ is specified in the sequel, the ordering refers to the pointwise ordering induced by the lattice operations, i.e.,

$$x \le y \iff x_i \le y_i \text{ for all } i \in \mathcal{N}.$$

2.2 Economic framework

Consider an economy populated by n nodes. Each of these nodes is to be thought of a distinct economic entity, or "financial node" participating in the clearing network. Each such entity may have nominal liabilities to other entities in the system. These nominal liabilities represent the promised payments due to other nodes in the network. We represent this structure of liabilities with the $n \times n$ *nominal liabilities matrix* L, where L_{ij} represents the nominal liability of node i to node j. As the notion of nominal claims seems to imply, we assume that all nominal claims are nonnegative and that no node has a nominal claim against itself. In order to reflect this economic interpretation, we specify that the nominal liabilities matrix is non-negative and that all of the diagonal elements of the matrix equal 0; that is, we assume that $\forall i, j \in \mathcal{N}, L_{ij} \geq 0$ and that $\forall i, L_{ii} = 0$. Let $e_i \geq 0$ be the exogenous *operating income* received by node i from sources "outside" the clearing system. Operating income can be viewed as the cash flows thrown off by the real assets controlled by the node. A *financial system* is thus a pair (L, e), consisting of a nominal obligations matrix, L and an operating income vector e, satisfying the conditions given above.

Let p_i represent the total dollar payment by node i to the other nodes in the system. Let $p = (p_1, p_2, \ldots, p_n)$ represent the vector of total payments made by the nodes. Let $\bar{p}_i$ represent total nominal obligation of i to all other nodes, that is,

$$\bar{p}_i = \sum_{j=1}^{n} L_{ij}. \tag{1}$$

Let $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$ represent the associated vector, which we will term the *total obligation vector*. This vector represents the payment level required for complete satisfaction of all contractual liabilities by all nodes. Let

$$\Pi_{ij} \equiv \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

and let Π represent the corresponding matrix, which we will term the *relative liabilities matrix*. This matrix captures the nominal liability of one node to another in the system as a proportion of the debtor node's total liabilities. We assume that all debt claims have equal priority. This equality of priority implies that the payment made by node i to node j is given by $p_i \Pi_{ij}$. This implies that the

the total payments received by i are equal to $\sum_{j=1}^{n} \Pi_{ij}^{T} p_j$. Further, all payments are made to some node in the system and therefore,

$$\forall i, \ \sum_{j=1}^{n} \Pi_{ij} = 1,$$

or, in matrix notation,

$$\Pi \mathbf{1} = \mathbf{1},$$

an equality we will use later in the analysis.

The total cash flow to the owners of the equity of node i equals the sum of the payments received by other nodes plus the operating income less the payments made to i's creditors. This implies that the value of node i equity equals

$$\sum_{j=1}^{n} \Pi_{ij}^{T} p_j + e_i - p_i.$$

Note also that, by using (1) and (2), the financial system (L, e), where L is a nominal payments matrix and e is a vector of operating incomes, can be equivalently described by the corresponding triple $(\Pi, \bar{p}, e)$, where Π is a relative liabilities matrix, $\bar{p}$ is a total liability vector and, e is an operating income vector. We will this description of a financial system in the subsequent analysis.

Intuitively, a clearing payment vector for the financial system should represent a specification of the payments made by each of the nodes in the financial system that is consistent with the legal rules allocating cash flows among nodes and among holders of debt and equity. Three criteria which must be satisfied are (a) limited liability, which requires that the total payments made by a node must never exceed the cash flow available to the node, (b) the priority of debt claims, which requires that stockholders in the node receive no value until the node is able to completely pay off all of its outstanding liabilities, and (c) proportionality, which requires that if default by occurs, all claimant nodes are paid by the defaulting node in proportion to the size of their nominal claim on firm assets. These desiderata motivate the following definition.

Definition 1. A *clearing payment vector* for the financial system $(\Pi, \bar{p}, e)$ is a vector $p^* \in [\mathbf{0}, \bar{p}]$ that satisfies the following conditions:

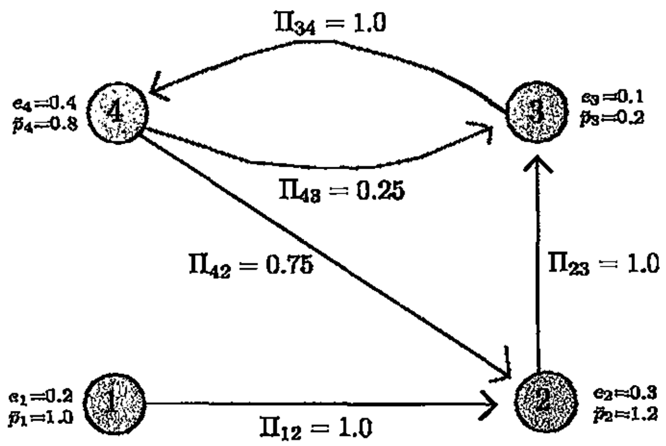

Figure 1. A Financial System

The above diagram depicts a financial system. The system consists of 4 nodes labeled 1, 2, 3, and 4. Beside each node is a record of the operating income it receives (e) and the total payments, $\bar{p}$, it is contracted to make with the other nodes in the system. The arrows between nodes indicate that the source node has an obligation to the target node. When such an obligation exists between two nodes, say i and j, the label Π_{ij}, denoting the proportion of i's total liabilities that are attributable to debts to j, is placed beside the arrow.

a. *Limited Liability.* $\forall i \in \mathcal{N}$,

$$p_i^* \leq \sum_{j=1}^{n} \Pi_{ij}^T p_j^* + e_i.$$

b. *Absolute Priority.* $\forall i \in \mathcal{N}$, either obligations are paid in full, that is, $p_i^* = \bar{p}_i$, or all cash flows are paid to creditors, that is,

$$p_i^* = \sum_{j=1}^{n} \Pi_{ij}^T p_j^* + e_i.$$

||

A clearing payment vector for the financial system illustrated in Figure 1 is provided by the vector $p^* = (0.20, 0.95, 0.20, 0.60)$. This vector calls for Node 1 to pay 0.20 to the other nodes. Because $\Pi_{12} = 1.0$, this payment is received entirely by node 2. Node 1 receives no inflows from other nodes in the clearing system; thus, Node 1's total inflows are given simply by its operating income of 0.20. Node 1's payment of 0.20 is less than its total obligations of 1.00. Consistent with absolute priority, the clearing vector thus requires Node 1 to pay out all of its cash flows. Node 2's payment under the clearing payment vector is 0.95, which is less than 2's obligated payment of 1.20.

Because $\Pi_{23} = 1$, this payment is received entirely by Node 3. Node 2 receives inflows both from Node 1 and from Node 4. The clearing vector calls for Node 4 to pay Node 2 $p_4^*\Pi_{24} = (0.60)(0.75) = 0.45$; as explained above, Node 1 pays Node 2 0.20. Thus, for Node 2, the total inflows from other nodes plus operating income equal $0.45 + 0.20 + 0.30 = 0.95$. Again, consistent with absolute priority, all of Node 2's inflows are paid out to creditors. Node 3's payment under the clearing payment vector is 0.20, which is equal to Node 3's obligated payment. Because $\Pi_{34} = 1$, this payment is received entirely by Node 4. Node 3 receives inflows both from Node 2 and Node 4. The inflow from Node 2 equals 0.95, the payment made by Node 2 under the equilibrium clearing payment vector. The inflow from 4 equals 0.15, 0.25 of 4's clearing payment of 0.60. Thus, the value of Node 3's equity is $0.10 + 0.95 + 0.15 - 0.20 = 1.00$. Node 2 receives a payment of 0.60 from Node 4,less than Node 4's obligated payment of 0.80, 0.15 of this payment goes to Node 3 and the remainder of 0.45 goes to Node 2. This payment exactly equals 4's cash inflow, which consists of operating income of 0.40 and a payment of 0.20 from Node 3. Note that the financial system being modeled is conservative in that wealth is neither created nor destroyed by the clearing process. Rather, the clearing process serves to distribute the 1.00 in operating income, received by the financial system as a whole, across the nodes. In this case, the entire balance is distributed to Node 3.

2.3 Basic network architecture

Definition 2. A set $S \subset \mathcal{N}$ is a *surplus* set if no node in the set has any obligations to any node outside the set and the set has positive operating income, that is if $\forall(i, j) \in S \times S^c, \Pi_{ij} = 0$ and $\sum_{i \in S} e_i > 0$. ∥

Lemma 1. *If p^* is a clearing vector, then it is not possible for all nodes in a surplus set to have zero equity value.*

Proof. Suppose S is a surplus set. Let P_i^+ represent the sum of all of the payments received by a node $i \in S$ from nodes in S^c. By the definition of surplus set, nodes in S make no payments to nodes in S^c. Thus, if all nodes in S have zero equity value, it must be the case that

$$p_i = \sum_{j \in S} \Pi^T_{ij} p_j + e_i + P_i^+, \qquad \forall i \in S. \tag{3}$$

Summing the equations specified in (3) over $i \in S$ thus yields

$$\sum_{i \in S} p_i = \sum_{j \in S} \sum_{i \in S} \Pi^T_{ij} p_j + \sum_{i \in S} (P_i^+ + e_i). \tag{4}$$

Using the fact that S is a surplus set, we also have that

$$\sum_{i \in S} \Pi^T_{ij} = 1, \qquad \forall i \in S. \tag{5}$$

Expressions (4) and (5) imply that

$$0 = \sum_{i \in S} (P_i^+ + e_i),$$

contradicting our assumption that $\sum_{i \in S} e_i > 0$. □

Establishing existence and uniqueness of clearing vectors requires that we present simple facts about the "architecture" of the financial system. The existence of a positive liability connecting two nodes in the system provides a conduit through which the risk of the debtor node can be transferred to the creditor. If we abstract from the magnitude of these exposures, we are left with a description of the financial system as a directed graph in which each debtor is linked via a directed edge to each of his creditors. These ideas are formalized below.

Definition 3. The *financial structure* graph associated with the financial structure $(\Pi, \bar{p}, e)$ is the directed multigraph whose vertices are the nodes of the financial network, $\mathcal{N}$, and whose edges are defined by $i \to j \iff \Pi_{ij} > 0$.[1] ||

The direct liabilities of each node in the system are to the nodes to which the agent has contractual obligations. However, these direct links by no means exhaust the set of all nodes that

[1] The technical distinction between a directed graph and a directed multigraph is that in a directed graph, there is, at most, one directed edge connecting any two nodes. In a directed multigraph, any number of edges can connect nodes.

are affected by a node's default. Defaults cascade through the system, the default of a single node, reduces the inflows to its creditors, perhaps triggering the default of one of these creditors, and even, perhaps, defaults further downstream. How far downstream can the risk of a given node in the system travel? An upper bound on propagation is provided by the concept of the *risk orbit* of a node in the system. The risk orbit of a node is the set of all nodes which are connected to the given node through some directed path, however circuitous, through the system.

Definition 4. For each node $i \in \mathcal{N}$, define the risk orbit of node i, denoted by $o(i)$, as follows: $o(i) = \{j \in \mathcal{N}$: there exists a directed path from i to $j\}$. ||

In the financial system illustrated in Figure 1, the risk orbits of the nodes are given as follows:

$$o(1) = \{1,2,3,4\} \quad o(2) = \{2,3,4\}$$

$$o(3) = \{2,3,4\} \quad o(4) = \{2,3,4\}.$$

The strongest sort of systemic interdependency, from the qualitative point of view we are currently pursuing, is for every node to be in the risk orbit of every other node, that is, for the financial structure graph induced by the financial system $(\Pi, \bar{p}, e)$ to satisfy the following condition: $\forall(i,j) \in N \times N$, $i \in o(j)$. When this condition is satisfied we will say the financial system is *strongly interlinked*. When a financial system is strongly interlinked, shocks hitting any node in the system, can be passed, perhaps through some very indirect routes, to any other node in the system. Becasue, the financial system presented in Figure 1 is not strongly interlinked, shocks to Nodes 2, 3, and 4 cannot affect Node 1. However, simply introducing, say, some obligation of Node 4 to Node 1, would render the system strongly interlinked.

It would appear that, because they abstract from the magnitude of exposures, concepts such as strong connectedness and risk orbits are incapable of providing any useful characterization of clearing payment vectors for the system. This is not correct. In fact, a very simple property of risk orbits forms the basis for our proof of the uniqueness of the clearing payment vector.

Lemma 2. *Suppose that p^* is a clearing vector for $(\Pi, \bar{p}, e)$, Let $o(i)$ be a risk orbit that satisfies*

$\sum_{j\in o(i)} e_j > 0$. *Then under* p^* *at least one node of* i *has positive equity value, that is,*

$$\exists j \in o(i), \text{ such that } \bar{p}_j < (\Pi^T p^* + e)_j.$$

Proof. First note that $o(i)$ is a surplus set. To see this, note that if some node, say i' in $o(i)$ owed something to a node $j \in o(i)^c$, then, by appending to the directed path from i to i' the edge $i' \to j$, one could construct a directed path from i to j, contradicting the assumption that j is not in $o(i)$. Lemma 1 shows that every surplus set contains a node with positive equity value. □

The intuition underlying Lemma 2 is clear. No financial "shock" can be absorbed by a bankrupt node of the financial system. The shock must be transferred, initially perhaps to other bankrupt nodes, but ultimately through some directed path(s) through the system to a solvent node. In the example considered in Figure 1, the lemma implies, since Node 3 is the only solvent node, that all other nodes contain Node 3 within their risk orbits. This is indeed the case, as can be seen from the risk orbits computed above.

2.4 Existence/Uniqueness of a clearing payment vector

Limited liability and absolute priority imply that $p^* \in [\mathbf{0}, \bar{p}]$ is a clearing payment vector if and only if the following condition holds: $\forall i \in \mathcal{N}$,

$$p_i^* = \min\left[e_i + \sum_{j=1}^{n} \Pi_{ij}^T p_j^*,\ \bar{p}_i\right].$$

The clearing payment vector, p^*, is thus a fixed point of the map, $\Phi(\cdot; \Pi, \bar{p}, e) : [\mathbf{0}, \bar{p}] \to [\mathbf{0}, \bar{p}]$, defined by

$$\Phi(p; \Pi, \bar{p}, e) \equiv (\Pi^T p + e) \wedge \bar{p}.$$

An economic interpretation of Φ is that $\Phi(p)$ represents the total funds that will be applied to satisfy debt obligations, assuming that nodes receive inflows specified by p from their debt claims on other nodes. The basic properties of the Φ-map are recorded in the following lemma.

Lemma 3. *The map* Φ *is positive, increasing, concave, and nonexpansive.*

Proof. The assertions of positivity, monotonicity and concavity follow because Φ is the composition of the positive, increasing, affine map $q \to \Pi^T q + e$, and the positive, increasing, concave map $q \to q \wedge \bar{p}$. To show that the map is nonexpansive, first note that, for any three vectors x, y, and z, $\|x \wedge z - y \wedge z\|_1 \leq \|x - y\|_1$. This result implies that $\|\Phi(p) - \Phi(p')\|_1 = \|(\Pi^T p + e) \wedge \bar{p} - (\Pi^T p' + e) \wedge \bar{p}\|_1 \leq \|\Pi^T p - \Pi^T p'\|_1$. Next note that the column sums of Π^T all equal 1. This implies, from basic matrix algebra, that $|||\Pi^T|||_1 = 1$. Thus, $\|\Pi^T p - \Pi^T p'\|_1 \leq \|p - p'\|_1$, establishing the result. □

Each of the regularity properties of the Φ map has a fairly straightforward interpretation. The fact that the map is positive just says that as long as inflows from the obligations of other nodes are positive, the node will itself make positive payouts. Monotonicity reflects the positive interdependence of the links in the financial system. The larger the payout s node received from other nodes on their debts, the larger the payout the node can itself make to other nodes. Concavity implies that increasing "dispersion" in the magnitude of the variation in payments made across nodes reduces overall ability to pay. Nonexpansivness reflects the "stability" in the clearing system. An increase in the input vector to the Φ map never yields a change in the output vector that is larger in absolute magnitude than the change in the input. Instead, although individual components of the output vector may grow disproportionatly, the change in the overall output vector is no larger in magnitude than the change in the input vector.

The previous lemmas form the basis for the first important result of our analysis: a demonstration of the existence of a clearing payment vector associated with every financial system, and of the uniqueness of clearing vectors under a fairly weak additional restriction that we term "regularity."

Definition 5. *A financial system is regular if every risk orbit, $o(i)$, is a surplus set.*

Lemma 4. *The following conditions are each by themselves sufficient for a financial system to be regular: (i) all nodes have positive equity balances or, (ii) the system is strongly interconnected and at least one node has a positive cash balance.*

Proof. It follows directly from the definitions of surplus set and strong interconnection, that (i) and (ii), ensure regularity.

Intuitively, regularity means that any maximal connected subset of nodes of the financial system has some surplus to transfer among the nodes of the system. The corollary shows that this will be the case whenever the financial system is strongly interlinked or each node is endowed with some transferable surplus.

Theorem 1. *Corresponding to every financial system* $(\Pi, \bar{p}, e)$,

a. *there exists a greatest and least clearing payment vector,* p^+ *and* p^-.

b. *Under all clearing vectors, the value of the equity at each node of the financial system is the same, that is, if* p' *and* p'' *are any two clearing vectors,*

$$(\Pi^T(p') + e - \bar{p})^+ = (\Pi^T(p'') + e - \bar{p})^+.$$

c. *If the financial system is regular, the greatest and least clearing vector are the same, i.e.,* $p^+ = p^-$, *implying that the clearing vector is unique.*

Proof. Let FIX(Φ) represent the set of fixed points of Φ. Because Φ is increasing, $\Phi(\mathbf{0}) \geq \mathbf{0}$ and $\Phi(\bar{p}) \leq \bar{p}$, the Tarski fixed-point theorem (see, e.g., Zeidler (1986) Theorem 11.E) implies that FIX(Φ) is non-empty and, moreover, possesses a greatest and least element. Thus (a) is established.

To prove (b) let p' be any clearing vector. We will show that the value of equity is the same under p' and p^+. This is sufficient to establish (b).

To show that the value of equity is the same under p' and p^+, first note that Π^T is an increasing map, as is the map $x \to x \vee \mathbf{0} \equiv x^+$. Thus, we must have, because $p^+ \geq p'$, that

$$(\Pi^T(p^+) + e - \bar{p}) \vee \mathbf{0} \geq (\Pi^T(p') + e - \bar{p}) \vee \mathbf{0}.$$

Thus, if

$$(\Pi^T(p^+) + e - \bar{p}) \vee \mathbf{0} \neq (\Pi^T(p') + e - \bar{p}) \vee \mathbf{0},$$

then we would have that

$$(\Pi^T(p^+) + e - \bar{p}) \vee \mathbf{0} \underset{\neq}{\geq} (\Pi^T(p') + e - \bar{p}) \vee \mathbf{0}. \tag{6}$$

Because p^+ and p^- are both clearing vectors, it also must be the case that

$$(\Pi^T(p^+) + e - \bar{p}) \vee \mathbf{0} = \Pi^T(p^+) + e - p^+, \tag{7}$$

$$\Pi^T(p') + e - \bar{p}) \vee \mathbf{0} = \Pi^T(p') + e - p'. \tag{8}$$

Expressions (6), (7) and (8) imply that

$$\Pi^T(p^+) + e - p^+ \underset{\neq}{\geq} \Pi^T(p') + e - p'. \tag{9}$$

Now, note that $\Pi\,\mathbf{1} = \mathbf{1}$. This implies that

$$\mathbf{1} \cdot (\Pi^T(p^+) - p^+) = \mathbf{1} \cdot (\Pi^T(p') - p') = 0.$$

. Thus,

$$\mathbf{1} \cdot (\Pi^T(p^+) + e - p^+) = \mathbf{1} \cdot (\Pi^T(p') + e - p'). \tag{10}$$

However, (9) implies that

$$\mathbf{1} \cdot (\Pi^T(p^+) + e - p^+) > \mathbf{1} \cdot (\Pi^T(p') + e - p'). \tag{11}$$

The contradiction between expressions (10) and (11) establishes (b).

Two distinct clearing vectors producing the same equity values at all nodes is not possible if the financial system is regular. To see this, first note that because p^+ and p' are distinct clearing vectors and $p' \leq p^+$, and because, for all nodes i that have positive equity value $p_i^+ = p_i' = \bar{p}_i$, it must be the case that for some i with zero equity value, $p_i^+ > p_i'$. Regularity implies that the risk orbit of i is a surplus set. By Lemmas 1 and 2, there exists a directed path $i = i_0 \rightarrow i_1 \rightarrow \ldots \rightarrow i_{l-1} \rightarrow i_l = j$ with the property that the nodes $i_1 \ldots i_{l-1}$ are zero-equity-value nodes, and $i_l = j$ is a positive-equity-value node. Because all cashflows into zero-equity nodes are paid out, $i_{k-1} \to i_k$, and $p' \leq p^+$, it follows that

$$p'_{i_{k-1}} < p^+_{i_{k-1}} \Rightarrow p'_{i_k} < p^+_{i_k}.$$

Thus, $p'_i < p^+_i$ implies that $p'_{i_{l-1}} < p^+_{i_{n-1}}$. Because $i_{l-1} \to i_l = j$. It follows that the payments received by j are higher under p^+ than under p'. Since j has positive equity value under both clearing payment vectors, and the payments received by j from other nodes cannot be any smaller under p^+ than they are under p' (because $p' \leq p^+$), it must be the case that the value of j's equity is strictly higher under p^+ than it is under p^-. This contradicts (b), and this contradiction establishes that $p^- = p^+$, i.e., that (c) holds. □

Some intuition for the importance of regularity for the uniqueness result is provided by the following example. Suppose the system contains two nodes, 1 and 2, and each node has a zero operating income. Moreover, each node has nominal liabilities of 1.00 to the other node. In our notation we have that $e = (0,0)^T$, $\bar{p} = (1,1)$, and

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This system is not a regular financial system, because the single risk orbit of the system $\{1,2\}$ is not a surplus set. In this example, any vector of the form $p_t = t\,(1,1), t \in [0,1]$ is a clearing vector for the system. In contrast, if we modify the example by giving one cent to the first node by setting $e' = (0.01, 0)$, we see that the unique clearing vector is given by $p^* = (1.00, 1.00)$. The payment vectors p_t, $t < 1$, do not satisfy the absolute priority condition under given e' because they leave Node 1 with an equity balance of 0.01 despite the fact that Node 1 has not completely satisfied its nominal obligation to Node 2.

3 Characterizing clearing vectors

3.1 Sequence of defaults

In this section we show that the clearing vector can be viewed as the product of a simulated or "fictitious" default process. This process both permits the construction of a simple algorithm for identifying clearing vectors and produces a natural metric for measuring a node's systemic risk exposure. We call this simple algorithm the *fictitious default algorithm*. In essence, the idea behind the algorithm is straightforward. First determine each node's payout assuming that all other nodes satisfy their obligations. If, under the assumption that all nodes pay fully, it is in fact the case that all obligations are satisfied, then terminate the algorithm. If some nodes default even when all other nodes pay, try to obvious the system again, assuming that only these "first-order" defaults occur. If only first-order defaults occur under the new clearing vector, then terminate the algorithm. If second-order defaults occur, then try to clear again assuming only second order defaults occur, etc. It is clear that since there are only n nodes, this process must terminate after n iterations. The point at which a node defaults under the algorithm is a measure of the node's exposure to the systemic risks faced by the clearing system.

Describing the algorithm in detail and proving that it is effective requires that we develop some new concepts. Let $\bar{\mathcal{S}}$ be the set of supersolutions of the fixed-point operator Φ; that is, $\bar{\mathcal{S}} = \{p \in [0, \bar{p}] : \Phi(p) \leq p\}$. Note that, for any such supersolution, because total equity value is positive, it must be the case that at least one node does not default, i.e., it is not possible that $\Phi(p) < \bar{p}$. For each $p \in \bar{\mathcal{S}}$, let the default set under p, which we denote by $\mathbf{D}(p)$, be the set of nodes i, such that $\Phi(p)_i < \bar{p}_i$. By the earlier observation, $\mathbf{D}(p)$ cannot contain all nodes. Let $\Lambda(p)$ represent the $n \times n$ diagonal matrix defined as follows:

$$\Lambda(p)_{ij} = \begin{cases} 1 & i = j \text{ and } i \in \mathbf{D}(p) \\ 0 & \text{otherwise.} \end{cases}$$

$\Lambda(p)_{ij}$ is a diagonal matrix whose values along the equal 1 along the diagonal in those rows representing nodes not in default under p, and equal to 0 otherwise. Thus, when multiplied by other matrices or vectors, the Λ matrix converts the entries corresponding to the nondefaulting node to 0. The complementary matrix $I - \Lambda(p')$ converts entries corresponding to defaulting nodes to 0. For fixed $p' \in \bar{\mathbf{S}}$, define the map $p \rightarrow \mathrm{FF}_{p'}(p)$ as follows:

$$\mathrm{FF}_{p'}(p) \equiv \Lambda(p')\Big(\Pi^T(\Lambda(p')p + (I - \Lambda(p')\bar{p}) + e\Big) + (I - \Lambda(p'))\Big(\bar{p}\Big). \qquad \text{(FIX)}$$

This map, $\mathrm{FF}_{p'}(p)$, simply returns, for all nodes not defaulting under p', the required payment $\bar{p}$, and, for all other nodes, returns the node's value assuming that nondefaulting nodes under p' pay in full, and defaulting nodes under p' pay p. By our earlier result, Lemma 1, the default set is not a surplus set. Thus, $\Lambda(p)\,\Pi$ has a row sum that is less than 1, and no row sum exceeds 1, this, in turn, implies that $\mathrm{FF}_{p'}$ has a unique fixed point by standard input-output matrix results (Karlin, 1959, Theorem 8.3.2). Call this fixed point $\mathrm{f}(p')$. Note that only when p' is a supersolution can we be assured that $\mathrm{f}(p')$ is well defined. Next, define inductively the following sequence of payment vectors.

$$p^0 = \bar{p}; \quad p^j = \mathrm{f}(p^{j-1}). \tag{FDS}$$

We call this sequence of vectors the *fictitious default sequence*, and we call the process of producing these vectors the *fictitious default algorithm.*

Lemma 5. *The fictitious default algorithm stated in (FDS) produces a well-defined sequence of vectors, p^j. This sequence decreases to the clearing vector in at most n iterations of the algorithm.*

Proof. First, we show by induction that the fictitious default sequence is well defined and decreasing. To show this, we must show that for all p^j, p^j is a supersolution to Φ for all j and that the sequence (p^j) decreases. We establish this result by induction. When $j = 0$, these assertions are obvious. Next, suppose the assertions are true for p^k. Note that the definition of the Λ matrix implies that $\Lambda(p^k)p^k + (I - \Lambda(p^k))\,\bar{p} = p^k$. Because p_k is a supersolution to Φ, it must be the case that for all defaulting nodes i, $(\Pi p^k + e)_i \leq p^k_i$. This implies, combined with the definition of Λ, that $\Phi(p^k) = \mathrm{FF}_{p^k}(p^k)$. By the induction hypothesis, p^k is a supersolution to Φ. Therefore, p^k is a supersolution to FF_{p^k}. This fact implies that p^{k+1}, the fixed point of FF_{p^k}, is less than or equal to p_k. Because $p^{k+1} \leq p^k$, the set of nodes at which default occurs must be no smaller under p^k than under p^{k+1}. Now, if the set of nodes is the same, then $\Phi(p^{k+1}) = \mathrm{FF}_{p^k}(p^k)$, and which implies, because by definition p^{k+1} is a fixed point of $\mathrm{FF}_{p^k}(p^k)$, that p^{k+1} is a fixed point of Φ, and thus trivially a supersolution. If the set of defaulting nodes is larger under p^{k+1}, then some nodes that paid their obligations in full under p^k default under p^{k+1}, and the rest of the nodes either default under both payment vectors or under neither. Thus, for these nodes such that default occurs under p^{k+1} but not p^k, $\phi(p^{k+1})_i < p_i^{k+1}$. For all other nodes, the fixed point construction implies that $\phi(p^{k+1})_i = p_i^{k+1}$. Thus, we have that p^j is a supersolution to Φ and that (p_j) is a weakly decreasing sequence.

As shown in the previous paragraph, if the set of defaulting nodes is the same under both p^{j+1} and p^j, then (i) p^j is a fixed point of Φ, and (ii) the *sequence will remain constant* after p_{j+1}. If p^j fails to be a fixed point of the map Φ, then a node that did not default under p^j defaults under p^{j+1}. In this case, the number of defaulting nodes, specified in the next Λ matrix, will increase in the next iteration. Because there are only n nodes and at most $n-1$ can default in any supersolution, it must be the case that the payment vector produced by the algorithm ceases to change after at most n iterations. Because the sequence *is constant only at* fixed points, the clearing vector is attained in at most n iterations. $\square$

The fictitious default algorithm works as follows. First, start with a trial solution which specifies that all nodes pay their obligations in full. If all node are indeed able to satisfy their obligations assuming that other nodes meet theirs, then the algorithm terminates with a clearing vector. If some node defaults under the first trial solution, fix the payments of the nondefaulting nodes under the first trial solution at full repayment and solve the linear equations that equate inflows and payments for those nodes that defaulted under the first trial solution. This process generates a second trial solution. If no new defaults occur under the second trial solution, then the second trial solution is a clearing vector. However, in the second trial solution, the value of the nodes will be lower than at the first trial solution because inflows will be given a smaller payment vector. Thus, some nodes that did not default under the first clearing vector may default under the second. If defaults occur, then fix the payments of the nodes that did not default under the second trial solution at full repayment and solve for the payments of the remaining nodes, etc. Continued iteration of this procedure produces a series of payoff vectors that converge to the clearing vector. Because the trial solution only changes when a new default occurs, convergence must occur in at most n rounds. Note that for large networks, this procedure is much more efficient than the extensive procedure of solving the linear equations that define the clearing vector for all possible subsets of nodes, because this sort of extensive procedure requires solving up to 2^n sets of linear equations rather than at most the n linear equations that must be solved using the fictitious default algorithm. An example illustrating the fictitious default algorithm is provided below. The parameters of the financial system are given as follows:

Example: Fictituous default algorithm

Financial system:

$$\Pi = \begin{pmatrix} 0 & \frac{15}{16} & \frac{1}{16} & 0 \\ 0 & 0 & \frac{15}{16} & \frac{1}{16} \\ \frac{1}{16} & 0 & 0 & \frac{15}{16} \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}, \quad \bar{p} = \left(1, \frac{6}{5}, \frac{1}{5}, \frac{4}{5}\right), \quad e = \left(\frac{1}{5}, \frac{3}{10}, \frac{1}{10}, \frac{2}{5}\right).$$

Steps in fictitious default algorithm:

	Trial solution: p	Default set: $\mathbf{D}(p)$
p^0	$\left(1, \frac{6}{5}, \frac{1}{5}, \frac{4}{5}\right)$	$\{1, 4\}$
p^1	$\left(\frac{1}{5}, \frac{6}{5}, \frac{1}{5}, \frac{3}{5}\right)$	$\{1, 2, 4\}$
p^2	$\left(\frac{1}{5}, \frac{19}{20}, \frac{1}{5}, \frac{3}{5}\right)$	$\{1, 2, 4\}$

In the example of the fictitious default algorithm, the initial solution, p^0, is set to full repayment. At this solution, Nodes 1 and 4 default. The next solution, p^1, set to equal the solution to the linear equations that clear the system assuming that only Nodes 1 and 4 default. At p^1, in fact, Node 2 defaults in addition to Nodes 1 and 4. The next iteration solves for p^2, the clearing vector assuming that only Nodes 1, 2, and 4 default. Under p^2, in fact, as assumed, only Nodes 1, 2, and 4 do default. Thus, the default set does not change and the algorithm terminates, producing the clearing vector in two iterations.

In addition to being computationally efficient, the algorithm has an economic interpretation: The step in the algorithm at which a node is added to the defaulting set can be used as a measure of the node's financial health. Nodes that default under the first trial solution are fundamentally insolvent because they cannot survive even with no systemic risk exposure. Nodes that fail in the next wave are quite fragile in that they fail whenever fundamentally insolvent nodes fail. The third-order failures are triggered by the failure of fragile, but not fundamentally unsound nodes, etc. Thus, nodes are partitioned by the algorithm into solvent nodes and $1, 2 \ldots n-1$-th order failures. Thus, the algorithm, combined with Monte Carlo simulation of exogenous income of the nodes, e, can be used to construct a probability distribution over orders of default for each node associated with the given stochastic shock to exogenous income. This distribution could form the basis for a practical metric for systemic-risk exposure to nodes in a financial network.

3.2 Programming characterization

Next we will show that clearing payment vectors can be identified by solving almost any programming problem that places weight on maximizing payments by all nodes in the system subject to the limited liability condition. Formally stated, we associate with each financial system $(\Pi, \bar{p}, e)$, and each function $f : [0, \bar{p}] \to \Re$, the programming is problem

$$\mathbf{P}(\Pi, \bar{p}, e, f) \qquad \underset{p \in [0, \bar{p}]}{\text{Max}} \; f(p)$$

$$s.t. \qquad p \leq \Pi^T p + e.$$

The link between this programming problem and clearing payment vectors for the financial system is provided by the following lemma.

Lemma 6. *If f is strictly increasing, then any solution to $\mathbf{P}(\Pi, \bar{p}, e, f)$ is a clearing vector for the financial system.*

Proof. If p^* solves $\mathbf{P}(\Pi, \bar{p}, e, c)$, the the fact that p^* is a feasible solution to $\mathbf{P}(\Pi, \bar{p}, e, c)$ ensures that p^* satisfies the limited liability condition for a clearing payment vector. If absolute priority were not satisfied, say at node i, then it would be the case that $p_i^* < \bar{p}$ and

$$(\Pi^T p^* + e \quad p^*)_i > 0.$$

Consider the vector p_ϵ which is equal to p^* in all components except i, and, for i, is given by $p_i^* + \epsilon$ where ϵ is chosen sufficiently small to ensure that limited liability remains satisfied. Because

$$(\Pi^T p_\epsilon + e - p_\epsilon)_j - (\Pi^T p^* + e - p^*)_j = \epsilon \Pi_{ij} \geq 0,$$

p_ϵ is feasible. Because p_ϵ is at least equal to p^* in all its components and greater than p^* in one of its components, and f is strictly increasing, it must be the case that $f(p^*) < f(p_\epsilon)$, contradicting the suposition that p^* is a solution to $\mathbf{P}(\Pi, \bar{p}, e, f)$. □

Because clearing vectors are determined entirely by the limited liability and absolute priority conditions, it follows that these two conditions always produce payoff vectors that maximize the total extraction of payments from the nodes in the financial system. Because the clearing vector is unique in any regular financial system, the result also implies that in regular financial systems, all decision makers who prefer more payments to less will agree that the clearing vector maximizes their objectives. Thus, for example, whether one attempts to maximize cents on the dollar paid or total payments, or payments weighted by a biased weighting scheme that favors some nodes over others, the end result will be the same—the selection of the clearing vector. The above result shows also that, for a regular financial system, solving the programming problem given by $\mathbf{P}(\Pi, \bar{p}, e, f)$ for a suitably chosen function f, say a linear function with positive weighting constants, is a way of computing the clearing vector. In fact, this is exactly the approach the monetary authorities in Kuwait took to clearing the financial net after the crash of the al-Manakh market. Given the $n-1$-step convergence of the fictitious default algorithm discussed above, however, this programming approach may not be an efficient way of computing clearing vectors given that only one variable will be introduced into the basic solution on each pivot. Algorithms that exploit the economics of the problem, such as the fictitious default algorithm developed above, allow for the simultaneous introduction of many defaulting nodes in a single step.

4 The Comparative statics of the clearing system

The first question we will address is how this clearing payment vector changes with changes in the exogenous parameters of the model. We first consider the relationship between this clearing payment vector and the operating income received by the system e, while holding the nominal liability matrix L (or equivalently Π and $\bar{p}$) constant. The basic characterization of this relationship is provided below. In order to ensure that the clearing vector is unique and, thus, that comparative statics in the traditional sense are possible, we henceforth restrict our attention to regular financial systems.

Lemma 7. *The clearing payment vector is a concave, increasing function of operating income and the level of nominal liabilities, and is concave in the relative liabilities matrix. In other words, the function* $e \to FIX(\Phi(\cdot;\Pi,\bar{p},e))$, *and the function* $\bar{p} \to FIX(\Phi(\cdot;\Pi,\bar{p},e))$ *are concave, increasing, and nonexpansive; further, the function* $\Pi \to FIX(\Phi(\cdot;\Pi,\bar{p},e))$ *is concave.*

Proof. For the purposes of this proof, define the function $F\colon [0,\bar{p}] \times \Re^n_{++} \colon \to [0,\bar{p}]$ by $F(p,e) \equiv \Phi(p,e;\Pi,\bar{p})$. The clearing payment vector is given by the function $f\colon \Re^n_{++} \to [0,\bar{p}]$, defined by $f(e) = \mathrm{FIX}(F(\cdot,e))$. A theorem from Milgrom and Roberts (1984) shows that the fact that F is increasing in e (established in Lemma 3) implies that f is increasing. To see that f is concave and nonexpansive, define a sequence of functions, $\{f_n(e)\}_{n=0}^{\infty}$, inductively as follows:

$$f_n(e) = F(f_{n-1}(e),e), \quad f_0(e) \equiv 0.$$

For each fixed $e \in \Re_{++}$, $f_n(e)$ is just the nth iteration of the map $p \to \Phi(p;\Pi,\bar{p},e)$ function starting at the initial payment vector 0. Thus standard results on the convergence of iterates of monotone increasing operators show that $f_n(e) \uparrow f(e)$, for all e. Using the fact that F is nondecreasing, jointly concave in p and e, and nonexpansive, induction shows that, for all n, f_n is concave and nonexpansive. Thus, f is the pointwise limit of nonexpansive concave functions and thus concave and nonexpansive. The above argument establishes the claim of the lemma for the function $e \to \mathrm{FIX}(\Phi(\cdot;\Pi,\bar{p},e))$. The proof of the claim for $\bar{p} \to \mathrm{FIX}(\Phi(\cdot;\Pi,\bar{p},e))$ and $\Pi \to \mathrm{FIX}(\Phi(\cdot;\Pi,\bar{p},e)$ is identical and thus will be omitted. □

Note that in the standard single-period/single-firm financial model, the payment to debtholders equals $\min[\bar{p},e]$ where e is the firm's operating earnings and $\bar{p}$ is the level of the firm's nominal liabilities. Thus, the payment received by debtholders is a concave, increasing, nonexpansive function of the firm's operating income and the level of nominal liabilities. Lemma 7 shows that these qualitative features of the debt payments in single-firm settings are inherited by the debt payment vectors of multi-node clearing systems. This result has a number of direct implications. For example, concavity of the payment stream in operating income implies that increases in the riskiness of operating income, in the sense of second-order stochastic dominance under the market-pricing measure, will reduce the expected payments received by debtholders and thus lower the value of debt claims. However, such risk shifts will not lead unambiguously to increased equity values for the nodes in the system. The reason for this is simple. In our model, all debt claims are owned by some stockholder at some node of the system. This implies that increases in risk across the system have two effects. First, they raise the value of equity by lowering the value of the debt payments made by stockholders to other nodes. Second, the increased risk also lowers the value of the portfolio of debt securities held as assets by each stockholder. Thus, the effect of global risk increases is ambiguous. The concavity of the clearing payment vector in the relative liabilities matrix implies that payment structures that are nondiversified (each firm makes all payments to one other firm) produce smaller clearing vectors than systems featuring diversified clearing vectors (each firm has roughly equal obligations to all other nodes).

Next, note that all of our results can also be interpreted in terms of node value. To understand this, note that the terminal-date equity in a financial system is $\Pi^T p^* + e - p^*$, and the debt is $p^*(e)$, where p^* is the clearing vector for the financial system. Thus, the total terminal value of any node in the system is the value of debt plus the value of equity, or $\Pi^T p^* + e$. Total value of all nodes in the economy is thus just $\mathbf{1} \cdot (\Pi^T p^* + e) = \mathbf{1} \cdot (p^* + e)$, the sum of the value of equity and the value of all payments on liabilities under the equilibrium clearing vector. From this result we obtain the following simple corollary to Lemma 6.

Corollary. *Increase in the diversification of obligations among nodes, increases the aggregate value of nodes in the clearing system.*

Another straightforward, but nevertheless interesting consequence of Lemma 6, relates to the effect of income volatility on the aggregate value of nodes in the financial system. Since, in an arbitrage-free economy, the initial value of a nodes is just the discounted expectation of its terminal value under the equilibrium pricing measure, and because the function mapping income to node value, $e \to \Pi^T p^*(e) + e$, is concave, the following corollary to Lemma 6 is immediate.

Corollary. *Increases in the unsystematic volatility of exogenous shocks (operating income) to the financial system lower the initial value of all nodes in the system.*

Thus. node value is reduced by unsystematic economic volatility, even though, in our analysis their are no dissipative consequences of financial distress even when markets are perfect and frictionless. Volatility reduces the size of payments between nodes and this reduces the market value of nodes. Since, clearly in the frictionless market setup specified above, unsystematic volatility has no adverse welfare consequences, this result should be interpreted as a caution against interpreting the reduction in corporate value caused by unsystematic risk as reflecting either market imperfections or irrational asset pricing.

Next, we show that, in some sense, convex combinations of financial systems can never have default or payment rates inferior to the worse of the two or superior to the better of the two. In order to permit a precise formulation of this idea, let $p^*(\Pi, \bar{p}, e)$ be the clearing payment vector associated with an arbitrary financial system $(\Pi, \bar{p}, e)$; that is, $p^*(\Pi, \bar{p}, e) \equiv \text{FIX}[\Phi(\cdot; \bar{p}, e)]$. A *$\lambda$-convex combination* of the financial systems $(\Pi', \bar{p}', e')$ and $(\Pi'', \bar{p}'', e'')$ is the financial system, $(\Pi_\lambda, \bar{p}_\lambda, e_\lambda)$, defined by

$$(\Pi_\lambda, \bar{p}_\lambda, e_\lambda) = \lambda(\Pi', \bar{p}', e') + (1-\lambda)(\Pi'', \bar{p}'', e''), \lambda \in [0, 1],$$

Lemma 8. *Suppose that the financial system $(\Pi_\lambda, \bar{p}_\lambda, e_\lambda)$ is a λ-convex combination of the financial systems $(\Pi', \bar{p}', e')$ and $(\Pi'', \bar{p}'', e'')$, then the equilibrium clearing payment vectors of the financial systems (p^*) satisfy the following inequalities:*

$$p^*(\Pi', \bar{p}', e') \wedge p^*(\Pi'', \bar{p}'', e'') \le p^*(\Pi_\lambda, \bar{p}_\lambda, e_\lambda) \le p^*(\Pi', \bar{p}', e') \vee p^*(\Pi'', \bar{p}'', e'').$$

Proof. Note that, for all $i \in N$, the function $\lambda \to \Phi(p; \Pi_\lambda, \bar{p}_\lambda, e_\lambda)_i$ is linear, and therefore monotone. Thus we have that

$$\Phi(p;\Pi',\bar{p}',e') \wedge \Phi(p;\Pi'',\bar{p}',e'') \leq \Phi(p;\Pi_\lambda,\bar{p}_\lambda,e_\lambda) \leq \Phi(p;\Pi',\bar{p}',e') \vee \Phi(p;\Pi'',\bar{p}',e'').$$

Let

$$H^-(p) \equiv \Phi(p;\Pi',\bar{p}',e') \wedge \Phi(p;\Pi'',\bar{p}',e''), \quad H^+(p) \equiv \Phi(p;\Pi',\bar{p}',e') \vee \Phi(p;\Pi'',\bar{p}',e'').$$

Note that H^- and H^+ are monotone, increasing maps defined on $[0,\bar{p}]$ with fixed points in this order interval. If p^+ is a fixed point of H^+ and p^- is a fixed point of H^-, then the above inequality implies that

$$p^- \leq p^*(\Pi_\lambda, e_\lambda) \leq p^+.$$

Because $p^*(\Pi',\bar{p}',e') \vee p^*(\Pi'',\bar{p}'',e'')$ is a supersolution to H^+, i.e.,

$$p^+ \leq p^*(\Pi',\bar{p}',e') \vee p^*(\Pi'',\bar{p}'',e'').$$

Similarly, because $p^*(\Pi',\bar{p}',e') \wedge p^*(\Pi'',\bar{p}'',e'')$ is a subsolution to H^-,

$$p^- \geq p^*(\Pi',\bar{p}',e') \wedge p^*(\Pi'',\bar{p}'',e'').$$

The inequalities follow. □

Lemma 8 is a fairly weak result. However, a stronger characterization, such as a concavity result for financial systems, i.e., a result showing that convex combinations of systems yield higher payment rates than convex combinations of the payment vector of the two systems being combined, cannot be obtained. In fact, it is easy to construct counterexamples to this stronger characterization.[2] The failure of concavity occurs because the map $(\Pi, p) \to \Phi(p; \Pi, \bar{p}, e)$ is not concave, although it is concave in each of the variables, Π and p, separately.

[2] A numerical counterexample is available upon request.

5 Possible extensions and concluding remarks

In this paper, we provide conditions for the existence and uniqueness of a clearing vector for a complex financial system, analyze the properties of the clearing vector, and provide comparative statics describing the relationship between the clearing vector and underlying parameters of the financial system. This work represents a contribution to our understanding of the modeling of complex financial systems featuring cyclical obligations between the parties. However, it is only a first step in the development of a research program in this area. In fact, one of the virtues of our analysis is that it can be extended in many directions. Extensions fall into three broad categories: (i) utilizing the current model for valuation and risk analysis; (ii) dealing with more complex legal/institutional structures; and (iii) incorporating dynamics.

The simplest extension of the present analysis is to use the formulae developed in the paper to value financial claims and assess default probabilities for financial networks. Given a structure of liabilities, the value of the debt and equity claims for a fixed level of exogenous income at the terminal date is determined by our model. If we assume exogenous income follows a standard stochastic process between the initial date and the clearing date, then this stochastic process, combined with the terminal boundary conditions imposed by our model and standard risk-neutral valuation technology, can generate prices for the debt and equity of the nodes in the system (see, for example, Duffie (1992)). In addition, probabilities of default and default correlation can be easily be computed. In addition the distribution of cash flows to each of the nodes also can be computed and inverted to yield value-at-risk estimates.

Extending our results to allow for more complex legal and institutional structures is almost as transparent. For example, the nodes in the system could be allowed to hold intercorporate equity claims as well as intercorporate debt claims. In this case inflows would be augmented by equity as well as debt inflows. Because equity claims are linear, this extension would not complicate our analysissignificantly. Multiple priority classes could be accommodated by our framework. To accommodate multiple priority classes, we would utilize a sequential clearing procedure in which first a clearing vector for senior claims is found, then the residual value is treated as the exogenous equity in the system for the second clearing of the next highest priority claim, etc. Another important extension would be to allow for violations of absolute priority, a significant factor in corporate bankruptcies, though not in some of the financial network clearing systems adressed earlier. The key assumptions that drive most of our results are that creditor claims are continuous and increasing in the value of the node. If violations of absolute priority are the product of efficient multilateral bargaining, as assumed in much of the literature (e.g., Brown, 1996), then creditor claims are likely to have this property. In networks where there are substantial fixed costs of financial distress, continuity is lost and, for this reason, one would expect to obtain more opaque results: for example, the lack of a unique clearing vector even when mild regularity conditions, such as those used in this paper, are imposed.

The most difficult direction of extension would be to allow for more than one clearing date, and thus incorporate true dynamics. In principle the extension is straightforward and would proceed as follows. First, allow for intercorporate equity and assume that nodes that default at a given date become wholly owned by their creditors from that date forward. Next, allow all nodes to borrow from a node outside the system that itself is not subject to default risk. The outside node, or "central bank," would provide funds at a competitive rate. Thus, nodes would only default when, at the clearing vector, the value of future inflows is less than the value of liabilities. Using this motif, and backward induction, one could recursively solve for clearing vectors. Uncertainty could be introduced into this framework by recursively computing the expected value of future inflows in order to determine the current economic value of the node and thus solve the default problem for successively earlier periods. Of course, this sort of extension of our analysis, through the "curse of dynamic programming" would greatly increase the complexity of the analysis.

6 References

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